

Existence, proper Pareto reducibility, and connectedness in multi-objective optimization

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Abstract

This paper is divided to two parts. In the first part, we provide elementary proofs for some important results in multi-objective optimization. The given proofs are so simple and short in compared to the existing ones. Also, a Pareto reducibility result is extended from efficiency to proper efficiency. The second part is devoted to the relationships between nonemptiness, R_{\geq}^p -(semi)compactness, external stability and connectedness of the set of nondominated solutions in multi-objective optimization. Furthermore, it is shown that some assumption in an important result, concerning connectedness, is redundant and should be removed.

Keywords: *Multi-objective programming; External stability; Connectedness; Proper efficiency; Pareto reducibility.*

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1 Introduction

Multi-objective optimization refers to maximiaing/minimizing more than one objective functions over a feasible set. The image of the feasible set under the objective functions is called the image space, and it is usually denoted by Y . The set of minimals of Y is denoted by Y_N . Two basic and important questions in multi-objective optimization are asking about the conditions under which $Y_N \neq \emptyset$ and Y_N is externally stable (i.e. each dominated point of Y is dominated by a member of Y_N) [4, 13]. Another important result in multi-objective optimization is representing the set of (weak) efficient solutions of a multi-objective problem with respect to that of its subproblems [4, 7, 13]. This subject is called Pareto reducibility. The "Pareto reducibility" term was first used by Popovici [12]. In the first part of this paper, we provide elementary proofs for some important results concerning existence, external stability, and Pareto reducibility. The given proofs are so simple and short in compared to the existing ones, and specially these are suitable for teaching purposes. Also, a result about Pareto reducibility is extended from efficiency to proper efficiency.

Connectedness and R_{\geq}^p -(semi)compactness of Y_N are also two important notions in multi-objective optimization [4, 13]. The second part of the paper establishes the equivalence of nonemptiness, external stability, R_{\geq}^p -compactness, and R_{\geq}^p -semicompactness of Y_N under appropriate assumptions. Furthermore, it is shown that one of the assumptions of a well-known result concerning connectedness (established in [8]) is redundant and should be removed.

The preliminaries are given in Section 2 and the main results are presented in Sections 3 and 4.

2 Preliminaries

For two vectors $y^0, y^* \in R^p$, we use the following componentwise orders:

- $y^0 \leq y^*$ iff $y_j^0 \leq y_j^*$ for each j ;
- $y^0 \leq y^*$ iff $y^0 \leq y^*$ and $y^0 \neq y^*$;
- $y^0 < y^*$ iff $y_j^0 < y_j^*$ for each j .

Three orders \geq , \geq , and $>$ are defined analogously. Using the componentwise order \geq , the following cone is defined, which is called the natural ordering cone:

$$R_{\geq}^p = \{y \in R^p : y \geq 0\}.$$

Definition 2.1. Let $Y \subseteq R^p$.

- (i) $y^0 \in Y$ is called a nondominated point of Y if there does not exist $y \in Y$ such that $y \leq y^0$;
- (ii) $y^0 \in Y$ is called a weakly nondominated point of Y if there does not exist $y \in Y$ such that $y < y^0$.

The set of all nondominated points and the set of all weakly nondominated points of Y are denoted by Y_N and Y_{WN} , respectively.

Definition 2.2. [5] Let $Y \subseteq R^p$. The vector $y^0 \in Y$ is called a properly nondominated point of Y if $y^0 \in Y_N$ and there exists scalar $M > 0$ such that for each $y \in Y$ and $i \in \{1, 2, \dots, p\}$ satisfying $y_i < y_i^0$ there exists $j \in \{1, 2, \dots, p\}$ such that $y_j > y_j^0$ and

$$\frac{y_i^0 - y_i}{y_j - y_j^0} \leq M.$$

The set of all properly nondominated points of Y is denoted by Y_{PN} .

Definition 2.3. [4] $Y \subseteq R^p$ is called

- (i) R_{\geq}^p -convex if $Y + R_{\geq}^p$ is convex;
- (ii) R_{\geq}^p -closed if $Y + R_{\geq}^p$ is closed;
- (iii) R_{\geq}^p -compact if $(y - R_{\geq}^p) \cap Y$ is compact for every $y \in Y$;
- (v) R_{\geq}^p -semicompact if every open cover of Y of the form $\{(y^i - R_{\geq}^p)^c : y^i \in Y, i \in \mathcal{I}\}$ has a finite subcover.

Definition 2.4. [4] Let $Y \subseteq R^p$. The set Y_N is called externally stable if $Y \subseteq Y_N + R_{\geq}^p$.

The set $C \subseteq R^p$ is a cone if $\lambda C \subseteq C$ for each $\lambda \geq 0$. If furthermore, $C + C \subseteq C$, then it is said to be a convex cone. If $C \cap (-C) = \{0\}$, then it is called pointed. The cone C is called proper if it is nonempty, $C \neq \{0\}$, and $C \neq R^p$. The nonnegative and positive polar cones corresponding to C are defined as follows, respectively:

$$C^+ = \{d \in R^p : d^t x \geq 0, \forall x \in C\},$$

$$C^{++} = \{d \in R^p : d^t x > 0, \forall x \in C \setminus \{0\}\}.$$

Consider a multi-objective optimization problem (MOP) as follows:

$$\begin{aligned} \min f(x) &= (f_1(x), f_2(x), \dots, f_p(x)), \\ \text{s.t. } x &\in X, \end{aligned} \tag{1}$$

where $X \subseteq R^n$ is a nonempty set and f is a vector-valued function composed of $p \geq 2$ real-valued functions. The image of X under f is denoted by $Y := f(X) \subseteq R^p$ and is referred to image space.

Definition 2.5. A feasible solution $\hat{x} \in X$ is called

- (i) an efficient solution to MOP (1) if there is no $x \in X$ such that $f(x) \leq f(\hat{x})$;
- (ii) a weakly efficient solution to MOP (1) if there is no $x \in X$ such that $f(x) < f(\hat{x})$.

The set of all efficient solutions and the set of all weakly efficient solutions of MOP (1) are denoted by $X_E(f)$ and $X_{WE}(f)$, respectively.

In order to obtain efficient solutions with bounded trade-offs, Geoffrion [5] suggested restricting attention to efficient solutions that are proper in the sense of the following definition.

Definition 2.6. [5] A feasible solution $\hat{x} \in X$ is called a properly efficient solution to MOP (1) if it is efficient and there is a real number $M > 0$ such that for all $i \in \{1, 2, \dots, p\}$ and $x \in X$ satisfying $f_i(x) < f_i(\hat{x})$ there exists an index $j \in \{1, 2, \dots, p\}$ such that $f_j(x) > f_j(\hat{x})$ and

$$\frac{f_i(\hat{x}) - f_i(x)}{f_j(x) - f_j(\hat{x})} \leq M.$$

The set of all properly efficient solutions of MOP (1) is denoted by $X_{PE}(f)$.

Let $\rho \subseteq \{1, 2, \dots, p\}$ be nonempty. The set of all efficient (resp. weakly efficient) solutions of

$$\begin{aligned} \min \quad & f^\rho(x) = (f_i(x); \quad i \in \rho) \\ \text{s.t.} \quad & x \in X, \end{aligned} \tag{2}$$

is denoted by $X_E(f^\rho)$ (resp. $X_{WE}(f^\rho)$). The set of all properly efficient solutions of Problem (2) is denoted by $X_{PE}(f^\rho)$.

3 Some elementary proofs

This section contains elementary proofs for three important results in multi-objective optimization theory.

The following theorem has been proved by Borwein [3] for general real linear vector spaces, and has been addressed in some reference books, including [4, 13], for finite dimensional multi-objective optimization. The proof addressed in [4, 13] is a technical proof utilizing the Zorn's lemma. In this paper, we present a simple and short proof for this result. The new proof is more appropriate for teaching purposes in compared to the existing one.

Theorem 3.1. [4, 13] Let $Y \subseteq R^p$ and $y^0 \in Y$. If $(y^0 - R_{\geq}^p) \cap Y$ is compact, then $Y_N \neq \emptyset$.

Proof. Consider the following auxiliary optimization problem:

$$\begin{aligned} \min \quad & \sum_{j=1}^p y_j \\ \text{s.t.} \quad & y \leq y^0, \\ & y \in Y. \end{aligned} \tag{3}$$

The set of feasible solutions of Problem (3) is $(y^0 - R_{\geq}^p) \cap Y$ which is compact. The objective function of this problem is continuous. Therefore, Problem (3) has an optimal solution, say y^* . We show that $y^* \in Y_N$. If $y^* \notin Y_N$, then

$$\exists \bar{y} \in Y; \quad \bar{y} \leq y^* \leq y^0, \quad \bar{y} \neq y^*.$$

Therefore, \bar{y} is a feasible solution to (3) and

$$\sum_{j=1}^p \bar{y}_j < \sum_{j=1}^p y_j^*.$$

This contradicts the optimality of y^* for (3) and completes the proof. \square

Remark 3.1. Although in this paper we considered the natural cone R_{\geq}^p for ordering R^p , the above result is still valid if one considers any pointed convex closed proper cone $C \subseteq R^p$ in lieu of R_{\geq}^p (as can be found in [11]). It can be proved utilizing our simple approach as well. To this end, consider $d \in C^{++}$ and construct the following auxiliary problem¹:

$$\begin{aligned} \min \quad & d^t y \\ \text{s.t.} \quad & y^0 - y \in C, \\ & y \in Y. \end{aligned} \tag{4}$$

¹It is not difficult to see that $C^{++} \neq \emptyset$; see Remark 1.6 and Proposition 1.10 in [11].

The set of feasible solutions of Problem (4) is the compact set $(y^0 - C) \cap Y$ and its objective function is continuous. Therefore, Problem (4) has an optimal solution, say y^* . If $y^* \notin Y_N$, then

$$\begin{aligned} \exists \bar{y} \in Y; \quad y^* - \bar{y} \in C \setminus \{0\} \\ \implies d^t(y^* - \bar{y}) > 0 \implies d^t y^* > d^t \bar{y}. \end{aligned}$$

On the other hand,

$$y^0 - \bar{y} = y^0 - y^* + y^* - \bar{y} \in C + C = C.$$

Hence, \bar{y} is a feasible solution to Problem (4) and contradicts the optimality of y^* .

The theorem below has been addressed in some reference books, including [4, 13], on finite dimensional multi-objective optimization. In the following, we present a simple and short proof for this result.

Theorem 3.2. [4, 13] Let $Y \subseteq R^p$ be nonempty and R_{\geq}^p -compact. Then Y_N is externally stable.

Proof. Consider arbitrary $y^0 \in Y$ and the auxiliary optimization problem (3) as defined in the proof of Theorem 3.1. Similar to the proof of Theorem 3.1, it can be seen that Problem (3) has an optimal solution, say $y^* \in Y_N$. Hence $y^* \in y^0 - R_{\geq}^p$. This implies

$$y^0 \in y^* + R_{\geq}^p \subseteq Y_N + R_{\geq}^p.$$

Thus,

$$Y \subseteq Y_N + R_{\geq}^p,$$

because $y^0 \in Y$ was arbitrary. This completes the proof. \square

Remark 3.2. The above result can be proved in a similar way if one considers any pointed convex closed proper cone $C \subseteq R^p$ instead of R_{\geq}^p . See Remark 3.1 for more detail.

The rest of this section is devoted to Pareto reducibility [12]. The following theorem gives a simple proof for Proposition 2.35 in [4] (see also Malivert and Boissard [7] and Lowe et al [10]). Malivert and Boissard [7] proved this result for representing the weak efficient set of (MOP) with respect to the efficient set of its subproblems. The proof given in [4, 7] is very complicated in compared to that given in the present paper.

Theorem 3.3. Assume that f_1, f_2, \dots, f_k are convex functions and the feasible set, X , is convex. Then

$$X_{WE}(f) = \bigcup_{\substack{\rho \subseteq \{1, 2, \dots, p\} \\ \rho \neq \emptyset}} X_E(f^\rho).$$

Proof. \supseteq : The proof of this part is trivial.

\subseteq : The convexity assumptions imply that $Y = f(X)$ is R_{\geq}^p -convex. Assume that $\hat{x} \in X_{WE}(f)$. By the weight-sum scalarization technique (Theorem 3.5 in [4]), there exists a nonzero vector $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_p) \geq 0$ such that \hat{x} is an optimal solution to

$$\begin{aligned} \min \quad & \Lambda^t f(x) \\ \text{s.t.} \quad & x \in X. \end{aligned}$$

Setting $\rho = \{i \in \{1, 2, \dots, p\} : \lambda_i > 0\}$, we have $\hat{x} \in X_{PE}(f^\rho) \subseteq X_E(f^\rho)$ according to Theorem 3.11 in [4]. This completes the proof. \square

In fact, in the proof of the above theorem, we proved the following theorem as well. The following theorem extends Theorem 2.36 in [4]. The equality provided in Theorem 3.4 is stronger than that given in Theorem 3.3 in the present paper and Theorem 2.36 in [4]. Notice that this equality provides a representation for weak efficient solutions with respect to the properly efficient solutions.

Theorem 3.4. Assume that f_1, f_2, \dots, f_k are convex functions and the feasible set, X , is convex. Then

$$X_{WE}(f) = \bigcup_{\rho \subseteq \{1, 2, \dots, p\}} X_{PE}(f^\rho). \quad (5)$$

The function $f : X \rightarrow R^p$ is called convexlike if for each $x, y \in X$ and each $\lambda \in (0, 1)$ there exists some $z \in X$ such that $f(z) \leq \lambda f(x) + (1 - \lambda)f(y)$. It is not difficult to see that; if f is convex-like, then $Y = f(X)$ is R_{\geq}^p -convex. The following result shows that Equation (5) holds if one replaces the assumption “ f is convex” with weaker assumption “ f is convexlike”. The proof of this theorem is similar to that of Theorem 3.3 and is hence omitted.

Theorem 3.5. *If f is a convex-like function on X , then*

$$X_{WE}(f) = \bigcup_{\emptyset \neq \rho \subseteq \{1, \dots, p\}} X_{PE}(f^\rho).$$

4 External stability and connectedness

The following theorem shows that the result given in Theorem 3.2 is still valid under weaker assumption “ R_{\geq}^p -semicompactness” in lieu of “ R_{\geq}^p -compactness”.

Theorem 4.1. *If $Y \subseteq R^p$ is nonempty and R_{\geq}^p -semicompact, then Y_N is externally stable.*

Proof. Let $y^0 \in Y$ be arbitrary. Setting

$$Y^0 = (y^0 - R_{\geq}^p) \cap Y,$$

we show that Y^0 is R_{\geq}^p -semicompact. To this end, assume that

$$Y^0 \subseteq \bigcup_{i \in I} (y^i - R_{\geq}^p)^c.$$

Then

$$Y \subseteq \left(\bigcup_{i \in I} (y^i - R_{\geq}^p)^c \right) \cup (y^0 - R_{\geq}^p)^c.$$

By assumption of the theorem, there exists $m \in N$ such that

$$Y \subseteq \left(\bigcup_{i=1}^m (y^i - R_{\geq}^p)^c \right) \cup (y^0 - R_{\geq}^p)^c.$$

Hence,

$$Y^0 = (y^0 - R_{\geq}^p) \cap Y \subseteq \bigcup_{i=1}^m (y^i - R_{\geq}^p)^c.$$

Therefore, Y^0 is R_{\geq}^p -semicompact.

Since Y^0 is R_{\geq}^p -semicompact, by Theorem 2.12 in [4], $Y_N^0 \neq \emptyset$. Thus, there exists $\bar{y} \in Y_N^0$. Now, we show that $\bar{y} \in Y_N$; otherwise

$$\exists y^* \in Y; \quad y^* \leq \bar{y} \leq y^0, \quad y^* \neq \bar{y}.$$

$$\implies \exists y^* \in Y^0, \quad y^* \leq \bar{y}, \quad y^* \neq \bar{y}.$$

These contradict $\bar{y} \in Y_N^0$. Therefore, $\bar{y} \in Y_N \cap Y_N^0$. Hence,

$$y^0 \in \bar{y} + R_{\geq}^p \subseteq Y_N + R_{\geq}^p.$$

Since $y^0 \in Y$ is arbitrary, we have

$$Y \subseteq Y_N + R_{\geq}^p,$$

and the proof is completed. \square

For a given set $Y \subseteq R^p$, the set Y^∞ is defined as follows:

$$Y^\infty := \{d \in R^p : d \neq 0, y + \alpha d \in Y, \forall y \in Y, \forall \alpha > 0\}.$$

In fact, Y^∞ is the set of recession directions of Y .

Lemma 4.1. [9] *Assume that $Y \subseteq R^p$ is a closed convex set. If there exist $\hat{y} \in Y$ and $d \in R^p$ such that $\hat{y} + \alpha d \in Y$ for any $\alpha > 0$, then $d \in Y^\infty$.*

Lemma 4.2. [9] Assume that $Y \subseteq R^p$ is a closed convex set. Then Y is unbounded if and only if $Y^\infty \neq \emptyset$.

The following theorem gives a full connection between the notions addressed in Definition 2.3.

Theorem 4.2. Let $Y \subseteq R^p$ be nonempty, R_{\geq}^p -convex and R_{\geq}^p -closed. If $(y - R_{\geq}^p) \cap Y$ is closed for every $y \in Y$, then the following statements are equivalent:

- i) $Y_N \neq \emptyset$.
- ii) $(y - R_{\geq}^p) \cap Y$ is bounded for every $y \in Y$.
- iii) Y is R_{\geq}^p -compact.
- iv) Y is R_{\geq}^p -semicompact.
- v) Y_N is externally stable.

Proof. (i \Rightarrow ii): By contradiction assume that there exists $\hat{y} \in Y$ such that $(\hat{y} - R_{\geq}^p) \cap Y$ is not bounded; thus $(\hat{y} - R_{\geq}^p) \cap (Y + R_{\geq}^p)$ is an unbounded closed convex set. Hence, by Lemma 4.2 there exists nonzero vector $d \in R^p$ such that

$$\hat{y} + \alpha d \in (\hat{y} - R_{\geq}^p) \cap (Y + R_{\geq}^p), \quad \forall \alpha > 0.$$

Thus $0 \neq d \leq 0$ which results $y + d \leq y$ and $y \neq y + d$ for each $y \in Y + R_{\geq}^p$. Furthermore, by Lemma 4.1,

$$y + d \in Y + R_{\geq}^p, \quad \forall y \in Y + R_{\geq}^p.$$

These imply $(Y + R_{\geq}^p)_N = \emptyset$. Therefore $Y_N = \emptyset$, due to the equality of $(Y + R_{\geq}^p)_N$ and Y_N . This contradicts the assumption.

(ii \Rightarrow iii): Trivial.

(iii \Rightarrow iv): See Proposition 2.14 in [4].

(iv \Rightarrow v): This part results from Theorem 4.1.

(v \Rightarrow i): Trivial. □

Notice that some assumptions of the above theorem are redundant in some parts. Although all assumptions are necessary in proving (i) \Rightarrow (ii), " R_{\geq}^p -convexity and R_{\geq}^p -closedness" are redundant in proving (ii) \Rightarrow (iii). To prove (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (i), three assumptions " R_{\geq}^p -convexity, R_{\geq}^p -closedness, and the closedness of $(y - R_{\geq}^p) \cap Y$ " are redundant.

Now, we deal with a topological property of the nondominated set, connectedness property. This property may help to explore the nondominated set starting from a single nondominated point using local search ideas. Connectedness will also make the task of selecting a final compromise solution among the set of nondominated solutions easier [4].

A set $S \subseteq R^p$ is called not connected if there exist open sets O_1, O_2 such that $S \subseteq O_1 \cup O_2$, $S \cap O_1 \neq \emptyset$, $S \cap O_2 \neq \emptyset$, and $S \cap O_1 \cap O_2 = \emptyset$. Otherwise, S is called connected.

One of the most important results concerning the connectedness of Y_N , has been proved by Naccache [8]. He established the following result.

Theorem 4.3. [8] If $Y \subseteq R^p$ is closed, convex, and R_{\geq}^p -compact, then Y_N is connected.

This result has been addressed in some text books, see e.g. Theorem 3.35 in [4]. Now, utilizing Theorem 4.2, we show that the " R_{\geq}^p -compactness assumption" in Theorem 4.3 results from other assumptions of this theorem, and hence it should be removed. If $Y_N = \emptyset$, then the result is trivial. So, we assume that $Y_N \neq \emptyset$. Since Y is closed and convex, Y satisfies all assumptions of Theorem 4.2. Therefore, Y is R_{\geq}^p -compact, due to Theorem 4.2.

Remark 4.1. From the above discussion, it can be seen that: If Y is closed and convex and $Y_N \neq \emptyset$, then all the following hold:

- (i) Y is R_{\geq}^p -compact,
- (ii) Y is R_{\geq}^p -semicompact,
- (iii) Y_N is externally stable,
- (iv) Y_N is connected.

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